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Small bound isomorphisms on the domain of a closed *- derivation in $C(K)$

Toshiko Matsumoto and Seiji Watanabe*

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In this article, it is shown that if there exists a small bound isomorphism between the domains of closed *-derivations, then the underlying compact Hausdorff spaces are homeomorphic.

Key Words : unbounded derivation, small bound isomorphism

1. Introduction

Let $C(K_i)$ be the Banach space of all complex valued continuous functions on a compact Hausdorff space K_i equipped with the supremum norm $\|\cdot\|_\infty$ $(i = 1, 2)$. The classical Banach-Stone theorem states that any surjective linear isometry between $C(K_1)$ and $C(K_2)$ is induced by a homeomorphism between K_1 and K_2. That is, if the spaces $C(K_1)$ and $C(K_2)$ are isometric, then K_1 and K_2 are homeomorphic. Amir and Cambern[1,2] showed that if there is a surjective linear isomorphism T from $C(K_1)$ to $C(K_2)$ such that $\|T\|_\infty\|T^{-1}\|_\infty < 2$, then K_1 and K_2 are homeomorphic. This is a very interesting generalization of the Banach-Stone theorem.

On the other hand, these results have also been extended to various other Banach spaces by many authors. For a compact subset X of the real line $\mathbb{R}$, we denote by $C^1(X)$ the Banach space of all continuously differentiable functions on X. Jarosz[3] proposed the following question : Is there a positive ε such that for any compact subsets X, Y of the real line $\mathbb{R}$, the Mazur distance $d(C^1(X),\ C^1(Y)) < 1 + \varepsilon$ implies that X and Y are homeomorphic? When the norm of $C^1(X)$ and $C^1(Y)$ are given by the c-norm, Cambern-Pathak[4] proved the existence of such ε under the additional assumption $\|T\|_\infty\|T^{-1}\|_\infty < \infty$. Jun and Lee[5] also obtained some partial answers for this question.

The purpose of this article is to study the above problem from the point of view of unbounded closed *-derivations.

We recall a closed *-derivation. A derivation δ in $C(K)$ is a linear mapping in $C(K)$ satisfying the following conditions :
(1) The domain $\mathfrak{D}(\delta)$ of δ is a norm dense subalgebra of $C(K)$.
(2) $\delta(fg) = \delta(f)g + f\delta(g)$ $(f, g \in \mathfrak{D}(\delta))$.
δ is said to be a *-derivation if it also satisfies:
(3) $f \in \mathfrak{D}(\delta)$ implies $f^* \in \mathfrak{D}(\delta)$ and $\delta(f^*) = \delta(f)^*$ where f^* means the complex conjugate of f.
δ is said to be closed if $f_n \in \mathfrak{D}(\delta)$, $f_n \to f$ and $\delta(f_n) \to g$ implies $f \in \mathfrak{D}(\delta)$ and $\delta(f) = g$, that is, δ is a closed linear operator. Several interesting properties of the domain of closed derivations have been investigated. For example, $\mathfrak{D}(\delta)$ contains the constant functions $\mathbb{C}1$, and C^1-functional

*Dept. of Mechanical and Control Engineering, Professor

calculus is possible in $\mathfrak{D}(\delta)$, that is, if $f(=f^*) \in \mathfrak{D}(\delta)$ and $h \in C^1([-\|f\|_\infty, \|f\|_\infty])$, then $h(f)(= h \circ f) \in \mathfrak{D}(\delta)$ and $\delta(h(f)) = h'(f)\delta(f)$ where h' means the derivative of h. For other properties of unbounded derivations in general C^*-algebras, we refer to 6),7).

The differentiation d/dt on the space $C^1(X)$ of continuously differentiable functions on a compact subset X of the real line $\mathbb{R}$ is a typical example of closed derivations. Therefore, we may regard the domain $\mathfrak{D}(\delta)$ of δ as a generalization of the Banach space $C^1(X)$. Moreover, if $\mathfrak{D}(\delta) = C(K)$, δ is bounded and $\delta \equiv 0$. These facts suggest a unified approach to deal with $C(K)$, $C^1(X)$ and several other spaces of differentiable functions together.

2. Isomorphisms of $\mathfrak{D}(\delta)$

Let K_i be a compact Hausdorff space, and let δ_i be a closed *-derivation in $C(K_i)$ $(i = 1, 2)$. Then $\mathfrak{D}(\delta_i)$ is a Banach space under the norm defined by $\|f\| := \|f\|_\infty + \|\delta_i(f)\|_\infty$. If there exists a surjective linear isometry from $\mathfrak{D}(\delta_1)$ to $\mathfrak{D}(\delta_2)$, then $K_1(\delta_1)$ and $K_2(\delta_2)$ are homeomorphic where $K_i(\delta_i) := \{x \in K_i : \exists f \in \mathfrak{D}_i(\delta_i) \text{ such that } \delta_i(f)(x) \neq 0\}$ $(i = 1, 2)$[8]. Moreover, we showed that any small bound surjective linear isomorphism between$\mathfrak{D}(\delta_1)$ and $\mathfrak{D}(\delta_2)$ induces a homeomorphism between $K_1(\delta_1)$ and $K_2(\delta_2)$[9].

Theorem *Let K_i be a compact Hausdorff space, and let δ_i be a closed $*$-derivation in $C(K_i)$ from $\mathfrak{D}(\delta_i)$ onto $C(K_i)$ with $Ker(\delta_i) = \mathbb{C}1$ $(i = 1, 2)$. Suppose that there exists $x_i \in K_i$ and $M_i(> 0)$ such that $\|f\|_\infty \leq M_i\|\delta_i(f)\|_\infty$ for $f \in \mathfrak{D}(\delta_i)$ with $f(x_i) = 0$ $(i = 1, 2)$. If T is a linear isomorphism from $\mathfrak{D}(\delta_1)$ onto $\mathfrak{D}(\delta_2)$ which satisfies*
(i) *if $\delta_1(f) \equiv 0$, then $\delta_2(Tf) \equiv 0$,*
(ii) $\|fg\| \leq \|TfTg\| \leq (1+\varepsilon)^2\|fg\|$,
(iii) $\|f\| \leq \|Tf\| \leq (1+\varepsilon)\|f\|$, *and*
(iv) $\varepsilon < \min\{\dfrac{57}{100(M_1+1)}, \dfrac{57}{100(M_2+1)}, \dfrac{1}{49}\}$,
then K_1 and K_2 are homeomorphic.

For the proof of this theorem, we need some lemmas.

Lemma 1 *Let K_i and δ_i be as in the theorem $(i = 1, 2)$. Let T be a linear map from $\mathfrak{D}(\delta_1)$ onto $\mathfrak{D}(\delta_2)$ which satisfies the assumption* (ii) *in the theorem. If $\|f\|_\infty \leq k \leq 1$ and $\|f\| = 1$, then $\|Tf\|_\infty \leq (1+\varepsilon)\sqrt{-k^2+2k}$.*

Proof From the definition of the norm in $\mathfrak{D}(\delta_i)$, we have

$$\begin{aligned}
\|Tf\|_\infty^2 &= \|(Tf)^2\|_\infty \leq \|TfTf\| \leq (1+\varepsilon)^2\|f^2\| \\
&= (1+\varepsilon)^2(\|f^2\|_\infty + 2\|f\delta_1(f)\|_\infty) \\
&\leq (1+\varepsilon)^2\{\|f\|_\infty(\|f\|_\infty + 2\|\delta_1(f)\|_\infty)\} \\
&= (1+\varepsilon)^2\{\|f\|_\infty(\|f\|_\infty + 2(1-\|f\|_\infty))\} \\
&= (1+\varepsilon)^2\{-\|f\|_\infty^2 + 2\|f\|_\infty\} \\
&\leq (1+\varepsilon)^2(-k^2+2k).
\end{aligned}$$

Lemma 2 *Let K_i, δ_i, T be as in the theorem $(i = 1, 2)$. If $f \in \mathfrak{D}(\delta_1)$, then $\dfrac{5}{7}\|\delta_1(f)\|_\infty \leq \|\delta_2(Tf)\|_\infty$.*

Proof For arbitrary fixed $f \in \mathfrak{D}(\delta_1)$, put $f_0 := f - f(x_i)$. Since we may assume $\delta_1(f_0) \neq 0$, there exists $x_0 \in K_1$ such that $|\delta_1(f_0)(x_0)| > \|\delta_1(f_0)\|_\infty - \varepsilon'$ for any $\varepsilon'(\|\delta_1(f_0)\|_\infty > \varepsilon' > 0)$. For any $k(>0)$ such that $\sqrt{-k^2+2k} < \frac{5}{56}$, we take a function $h \in C^1([-\|f_0\|_\infty, \|f_0\|_\infty])$ such that

$$h'(f_0(x_0)) = 2, \quad 0 \leq h' \leq 2 \quad \text{and} \quad \|h\|_\infty \leq 2k(\|\delta_1(f_0)\|_\infty - \varepsilon').$$

Putting $g := h(f_0)$, we have

$$\begin{aligned} \|g\|_\infty = \|h\|_\infty &\leq 2k(\|\delta_1(f_0)\|_\infty - \varepsilon') \\ &< k|h'(f_0(x_0))||\delta_1(f_0)(x_0)| \\ &\leq k\|h'(f_0)\delta_1(f_0)\|_\infty = k\|\delta_1(g)\|_\infty. \end{aligned}$$

Hence,

$$\|g\|_\infty < k\|\delta_1(g)\|_\infty \qquad (1) \quad \text{and} \quad 2(\|\delta_1(f_0)\| - \varepsilon') < \|\delta_1(g)\|_\infty. \qquad (2)$$

We now show that

$$2\|\delta_1(f_0)\|_\infty(1 - (1+\varepsilon)\sqrt{-k^2+2k}) - 2\varepsilon' \leq \|\delta_2(Tg)\|_\infty. \qquad (3)$$

By (1) and Lemma 1,

$$\|Tg\|_\infty \leq \|g\|(1+\varepsilon)\sqrt{-k^2+2k}.$$

From this and $\|\delta_1(g)\|_\infty = \|h'(f_0)\delta_1(f_0)\|_\infty \leq 2\|\delta_1(f_0)\|_\infty$, we have

$$\begin{aligned} \|Tg\|_\infty - \|g\|_\infty(1+\varepsilon)\sqrt{-k^2+2k} &\leq \|\delta_1(g)\|_\infty(1+\varepsilon)\sqrt{-k^2+2k} \\ &\leq 2(1+\varepsilon)\sqrt{-k^2+2k}\|\delta_1(f_0)\|_\infty. \qquad (4) \end{aligned}$$

By $(1+\varepsilon)\sqrt{-k^2+2k} < 1$ and (iii), we have

$$\|g\|_\infty(1+\varepsilon)\sqrt{-k^2+2k} + \|\delta_1(g)\|_\infty < \|g\|_\infty + \|\delta_1(g)\|_\infty = \|g\| \leq \|Tg\|,$$

that is,

$$\|\delta_1(g)\|_\infty \leq \|Tg\| - \|g\|_\infty(1+\varepsilon)\sqrt{-k^2+2k}.$$

From this and (2), (4), we have

$$\begin{aligned} 2(\|\delta_1(f_0)\|_\infty - \varepsilon') &\leq \|Tg\| - \|g\|_\infty(1+\varepsilon)\sqrt{-k^2+2k} \\ &\leq \|\delta_2(Tg)\|_\infty + 2(1+\varepsilon)\sqrt{-k^2+2k}\|\delta_1(f_0)\|_\infty, \end{aligned}$$

which implies (3).

Since

$$\begin{aligned} \|Tg\|_\infty &\leq \|g\|(1+\varepsilon)\sqrt{-k^2+2k} \\ &< 2(1+k)\|\delta_1(f_0)\|_\infty(1+\varepsilon)\sqrt{-k^2+2k} \end{aligned}$$

from Lemma 1 and (1), we have

$$\begin{aligned} &\|T(f_0 - g)\| \\ \geq &\|Tf_0\|_\infty - \|Tg\|_\infty - \|\delta_2(Tf_0)\|_\infty + \|\delta_2(Tg)\|_\infty \\ \geq &\|Tf_0\|_\infty - 2(1+k)(1+\varepsilon)\sqrt{-k^2+2k}\|\delta_1(f_0)\|_\infty \\ &+ 2(1 - (1+\varepsilon)\sqrt{-k^2+2k})\|\delta_1(f_0)\|_\infty - 2\varepsilon' - \|\delta_2(Tf_0)\|_\infty. \qquad (5) \end{aligned}$$

Since $\|g\|_\infty = \|g\| - \|\delta_1(g)\|_\infty < 2(1+k)\|\delta_1(f_0)\|_\infty - 2(\|\delta_1(f_0)\|_\infty - \varepsilon')$, we have

$$\begin{aligned}\|T(f_0 - g)\| &\le (1+\varepsilon)\|f_0 - g\| \\ &\le (1+\varepsilon)(\|f_0\|_\infty + \|g\|_\infty + \sup_{x\in K_1} |\delta_1(f_0)(x)(1 - h'(f_0(x))|) \\ &\le (1+\varepsilon)(\|f_0\| + \|g\|_\infty) \\ &< \|f_0\| + \frac{57}{100}\|\delta_1(f_0)\|_\infty + (1+\varepsilon)(2k\|\delta_1(f_0)\|_\infty + 2\varepsilon'). \qquad (6)\end{aligned}$$

If, for some $k > 0$,

$$\|\delta_2(Tf_0)\|_\infty < \frac{5}{7}(1 - \frac{28}{5}(1+\varepsilon)\sqrt{-k^2+2k})\|\delta_1(f_0)\|_\infty,$$

then by (5)

$$\begin{aligned}&\|T(f_0 - g)\| \\ >&\|Tf_0\|_\infty - 2(1+k)(1+\varepsilon)\sqrt{-k^2+2k}\|\delta_1(f_0)\|_\infty + 2(1 - (1+\varepsilon)\sqrt{-k^2+2k})\|\delta_1(f_0)\|_\infty \\ &- 2\varepsilon' - \frac{5}{7}(1 - \frac{28}{5}(1+\varepsilon)\sqrt{-k^2+2k})\|\delta_1(f_0)\|_\infty \\ =&\|Tf_0\|_\infty + \{-2(1+k)(1+\varepsilon) + 2(1+\varepsilon)\}\sqrt{-k^2+2k}\|\delta_1(f_0)\|_\infty + (2 - \frac{5}{7})\|\delta_1(f_0)\|_\infty - 2\varepsilon' \\ =&\|Tf_0\|_\infty + \frac{4}{7}\|\delta_1(f_0)\|_\infty + \frac{5}{7}(1 - 2k(1+\varepsilon)\sqrt{-k^2+2k})\|\delta_1(f_0)\|_\infty - 2\varepsilon' \\ >&\|Tf_0\| + \frac{4}{7}\|\delta_1(f_0)\|_\infty - 2\varepsilon'.\end{aligned}$$

From this and (6), we see that

$$\|f_0\| + \frac{57}{100}\|\delta_1(f_0)\|_\infty + (1+\varepsilon)(2k\|\delta_1(f_0)\|_\infty + 2\varepsilon') > \|Tf_0\| + \frac{4}{7}\|\delta_1(f_0)\|_\infty - 2\varepsilon'.$$

Since k and ε' is arbitrary, we have

$$\|f_0\| \ge \|Tf_0\| + \frac{1}{700}\|\delta_1(f_0)\|_\infty.$$

This contradicts to (iii). Hence, for any sufficiently small $k > 0$, we have

$$\|\delta_2(Tf_0)\|_\infty \ge \frac{5}{7}(1 - (\frac{28}{5})(1+\varepsilon)\sqrt{-k^2+2k})\|\delta_1(f_0)\|_\infty,$$

which implies that

$$\|\delta_2(Tf_0)\|_\infty \ge \frac{5}{7}\|\delta_1(f_0)\|_\infty.$$

Since $\delta_1(f_0) = \delta_1(f)$, we have $\delta_2(Tf_0) = \delta_2(Tf)$ from (i). Consequently, we have

$$\|\delta_2(Tf)\|_\infty \ge \frac{5}{7}\|\delta_1(f)\|_\infty.$$

This completes the proof of Lemma 2.

Lemma 3 *Let K_i, δ_i and T be as in the theorem ($i = 1, 2$). If $f \in \mathfrak{D}(\delta_1)$, then* $\frac{5}{7}\|\delta_1(f)\|_\infty \le \|\delta_2(Tf)\|_\infty \le \frac{7}{5}(1+\varepsilon)\|\delta_1(f)\|\infty.$

Proof If $\delta_2(Tf) \equiv 0$, then $\delta_1(f) \equiv 0$ by Lemma 2. Replacing T by $(1+\varepsilon)T^{-1}$ and applying Lemma 1 and 2, we see that

$$\frac{5}{7}\|\delta_2(g)\|_\infty \leq (1+\varepsilon)\|\delta_1(T^{-1}(g))\|_\infty \quad \text{for } g \in \mathfrak{D}(\delta_2).$$

Since T is a onto map, we have

$$\frac{5}{7}\|\delta_2(Tf)\|_\infty \leq (1+\varepsilon)\|\delta_1(f)\|_\infty.$$

This completes the proof.

Proof of the theorem Fixed $x_0 \in K_1$. For $f \in C(K_1)$, there is unique function $F \in \mathfrak{D}(\delta_1)$ such that $\delta_1(F) = f$ and $F(x_0) = 0$. Let S be a linear map from $C(K_1)$ into $\mathfrak{D}(\delta_1)$ defined by $S(f) := F$ for $f \in C(K_1)$. Let S' be a map from $\mathfrak{D}(\delta_2)$ onto $C(K_2)$ defined by $S'(g) := \delta_2(g)$ for $g \in \mathfrak{D}(\delta_2)$. Since

$$\delta_1(S(\alpha f + \beta g) - \alpha S(f) - \beta S(g)) \equiv 0$$

for $f, g \in C(K_1)$, we have

$$\delta_2(TS(\alpha f + \beta g) - \alpha TS(f) - \beta TS(g)) \equiv 0$$

from (i), that is,

$$S'TS(\alpha f + \beta g) - \alpha S'TS(f) - \beta S'TS(g) \equiv 0.$$

Hence $S'TS$ is a linear map from $C(K_1)$ into $C(K_2)$. Let T' be a map from $C(K_1)$ to $C(K_2)$ defined by $T'(\delta_1(f)) := \delta_2(Tf)$ for $f \in \mathfrak{D}(\delta_1)$, then

$$T'(\delta_1(f)) = \delta_2(Tf) = \delta_2(TS(\delta_1(f)) = S'TS(\delta_1(f)).$$

Therefore, by Lemma 3, T' is a surjective linear map such that

$$\|T'\|_\infty \leq \frac{7}{5}(1+\varepsilon) \quad \text{and} \quad \|T'^{-1}\|_\infty \leq \frac{7}{5}.$$

Therefore

$$\|T'\|_\infty\|T'^{-1}\|_\infty \leq \frac{49}{25}(1+\varepsilon).$$

Then K_1 and K_2 are homeomorphic from the Amir theorem. This completes the proof of the theorem.

Applying the theorem to the differentiation, we have the following corollaries.

Corollary 1 *Let X and Y be compact subsets of the real line $\mathbb{R}$ with $X \subset [a, b]$ and $Y \subset [c, d]$. For $f_0 \in C([a, b])$ and $g_0 \in C([c, d])$ such that $f_0(x) \neq 0$ $(x \in [a, b])$ and $g_0(y) \neq 0$ $(y \in [c, d])$, set $\delta_1 := f_0\frac{d}{dt}$ and $\delta_2 := g_0\frac{d}{dt}$ respectively. If T is a linear isomorphism from $C^1(X)$ onto $C^1(Y)$ which satisfies*
(i) *if $\delta_1(f) \equiv 0$, then $\delta_2(Tf) \equiv 0$,*
(ii) $\|fg\| \leq \|TfTg\| \leq (1+\varepsilon)^2\|fg\|$,
(iii) $\|f\| \leq \|Tf\| \leq (1+\varepsilon)\|f\|$, *and*
(iv) $\varepsilon < \min\left\{\frac{57}{100\{(b-a)\|f_0\|_\infty + 1\}}, \frac{57}{100\{(d-c)\|g_0\|_\infty + 1\}}, \frac{1}{49}\right\}$,
then X and Y are homeomorphic.

Corollary 2[5] *Let X and Y be compact subsets of the real line $\mathbb{R}$ with $X \subset [a, b]$ and $Y \subset [c, d]$. If T is a linear isomorphism from $C^1(X)$ onto $C^1(Y)$ which satisfies*
(i) *if $f' \equiv 0$, then $(Tf)' \equiv 0$,*
(ii) $\|fg\| \leq \|TfTg\| \leq (1+\varepsilon)^2\|fg\|$,
(iii) $\|f\| \leq \|Tf\| \leq (1+\varepsilon)\|f\|$, *and*
(iv) $\varepsilon < \min\{\frac{57}{100(b-a+1)}, \frac{57}{100(d-c+1)}, \frac{1}{49}\}$,
then X and Y are homeomorphic.

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